# How Accurately Can the Method of Fundamental Solutions Solve the Inverse Problem of Electrocardiology?

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## Abstract

*This study presents a detailed comparison between the Method of Fundamental Solutions (MFS) approach to solving the inverse problem of electrocardiology and a more conventional boundary element method (BEM) approach.*

*Synthetic data were created to simulate the heart surface potential distribution during the time course of normal and ectopic heart beats. Both measurement and geometry noise were added to the data and the inverse problem was solved via both methods. Under these conditions several regularisation parameter determination methods were compared, with the Robust Generalised Cross-Validation (RGCV) method consistently performing better than any other method for both MFS and BEM approaches.*

*The MFS approach to solving the inverse problem of electrocardiology can sometimes yield more accurate results than the BEM approach, especially when the regularisation parameter is determined by RGCV, but BEM is generally superior.*

## 1. Introduction

To solve the inverse problem of electrocardiology, we first need to solve the forward problem of electrocardiology. That is, we solve Laplace's equation for the electric potential in the torso, $\phi$,

$$\nabla \cdot \sigma \nabla \phi = 0, \quad \mathbf{x} \in \Omega \tag{1}$$

subject to a given potential distribution $h(\mathbf{x})$ on the heart surface,

$$\phi = h(\mathbf{x}) \quad \mathbf{x} \in \Gamma_h, \tag{2}$$

and assuming the body surface is insulated,

$$(\sigma \nabla \phi) \cdot \mathbf{n} = 0 \quad \mathbf{x} \in \Gamma_T, \tag{3}$$

where $\Gamma_h$ and $\Gamma_T$ are the heart and body surfaces, respectively, $\sigma$ is the conductivity tensor within the torso, $\Omega$, and $\mathbf{n}$ is the outward pointing surface normal.

Here, we will solve the forward problem via the boundary element method (BEM) [1] and the Method of Fundamental Solutions (MFS) [2]. Both solution approaches reduce the forward problem to a matrix–vector equation. Concomitant solutions to the inverse problem will be compared using synthetic data generated during the QRS complex for normal and ectopic heart beats. Since the inverse problem is ill-posed for both solution methods, although in different ways, Tikhonov regularisation is required to determine the inverse solution. We also compare a range of regularisation parameter determination methods for obtaining the inverse solutions.

## 2. Methods

### 2.1. The Boundary Element Method

The BEM is a boundary method for solving Laplace's equation that exploits the equation's fundamental solution

$$f(r) = \frac{1}{4\pi r} \tag{4}$$

where $r = \|\mathbf{x} - \mathbf{y}\|$ represents the distance between some observation point $\mathbf{x}$ and some source point $\mathbf{y}$. By using Green's theorem and carefully handling the singular integrals that arise, the governing equation can be reduced to the matrix–vector equation

$$\mathbf{A}_B \mathbf{h} = \mathbf{b}_B \tag{5}$$

where $\mathbf{h}$ is a vector of heart surface potentials on the mesh points that represent an approximation to the heart surface, $\mathbf{b}_B$ is a vector of body surface potentials, measured at certain sites on the torso surface, and $\mathbf{A}_B$ is the forward transfer matrix [3]. Typically, the number of heart surface nodes, $M_H$, is less than the number of body surface nodes $M_T$; hence equation (5) represents an overdetermined system of linear algebraic equations.

### 2.2. The Method of Fundamental Solutions

The method of fundamental solutions (MFS) was introduced in the late 1970s to solve boundary value problems

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of the form given by equations (1), (2) and (3). In particular, it has recently been applied to solving the inverse problem of electrocardiology [4]. For this formulation, we have equation (1), along with the insulation boundary condition (3) and the new condition

$$\phi(\mathbf{x}) = m(\mathbf{x}) \qquad \mathbf{x} \in \Gamma_J \subset \Gamma_T \tag{6}$$

where $\Gamma_J$ represents points, say $N$ of them, where the potential $m(\mathbf{x})$ is measured (e.g. by a jacket). That is, both Dirichlet and Neumann conditions are specified on the one surface, and this is called a *Cauchy* Problem. Heart surface potentials are determined in a "post-processing" step.

MFS is also a boundary method, utilising the same fundamental solution (4), but avoids singular integrals by choosing a set of source points $\{\mathbf{y}_j\}$ lying outside the computational domain. Here, the source points will be created by pushing the torso mesh points "outside" the torso and shrinking the heart mesh points "inside" the heart. Solving Laplace's equation via MFS assumes a solution of the form

$$\phi(\mathbf{x}) = c_0 + \sum_{j=1}^{M} \frac{c_j}{4\pi\|\mathbf{x} - \mathbf{y}_j\|} \tag{7}$$

where $M = M_T + M_H$.

Enforcing the conditions (3) and (6) results in a $2N \times (M+1)$ system of algebraic equations for the coefficients [4] $c_j,\ j = 1, \ldots, M$,

$$\mathbf{A}_M \mathbf{c} = \mathbf{b}_M \tag{8}$$

where $\mathbf{c}$ is a vector representing the unknown coefficients, $\{c_j\}$ and $\mathbf{b}_M$ is vector representing values of the Dirichlet and Neumann conditions. Note that generally, $2N < M + 1$, so the system is underdetermined.

## 2.3. Regularisation

Both methods described above result in a non-square system of algebraic equations that must be solved. For BEM, solution of the system of equations gives the heart surface potentials, whereas for MFS, the system of equations gives a set of coefficients from which the heart surface potentials can be obtained [4]. Both situations can utilise zero-order Tikhonov regularisation to obtain a solution to the system of algebraic equations $\mathbf{A}\mathbf{x} = \mathbf{b}$:

$$\min_{\mathbf{x}} \left\{ \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 + \lambda^2 \|\mathbf{x}\|_2^2 \right\} \tag{9}$$

giving

$$\mathbf{x} = \left(\mathbf{A}^T\mathbf{A} + \lambda^2\mathbf{I}\right)^{-1} \mathbf{A}^T\mathbf{b}. \tag{10}$$

Here, $\lambda$ is a regularisation parameter and zero-order Tikhonov regularisation has been assumed. The difficulty with finding $\mathbf{x}$ is that $\lambda$ is unknown.

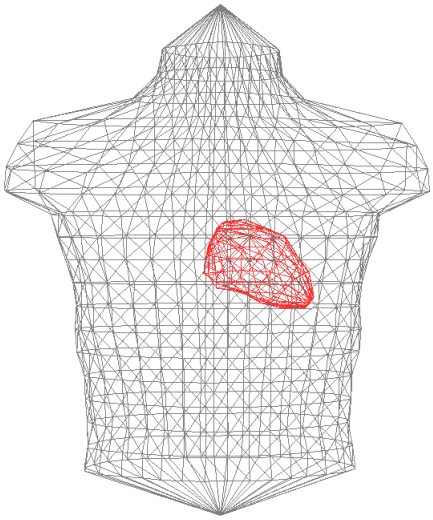


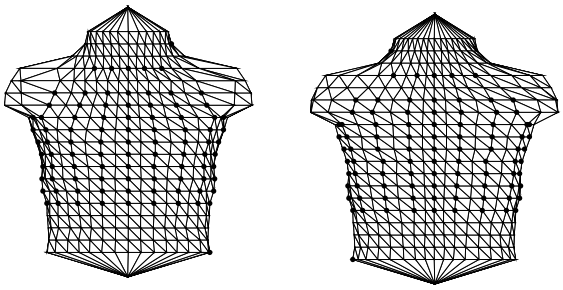
Figure 1. Combined heart (red) and torso mesh.

Several methods have been developed to find an "optimal" value for $\lambda$, all relying on using a singular value decomposition of the coefficient matrix $\mathbf{A}$ [5]. For this study we will compare the $L$-Curve Method [6], the Composite REsidual and Smoothing Operator (CRESO) method [7], the zero-crossing method (ZeroX) [8], the Generalised Cross-Validation (GCV) [9], and the Robust Generalised Cross-Validation Method (RGCV) [10]. Further details about each of these methods can be found in the review article [11].

## 2.4. Simulation Protocols

Detailed descriptions of the simulation protocols are given elsewhere [8, 11], but briefly the torso model (figure 1) presented in [12] was used with a cellular automata heart model [13] to produce a time course of the electric potential for normal and ectopic heart beats. Both "white" noise and geometry noise (by offsetting the heart) were incorporated into the simulated body surface potentials as recorded on a "jacket" (figure 2) and inverse solutions obtained. Simulations were performed at five different noise levels, eight time points through the heart beat, each with seven different heart offsets, giving a total of 280 scenarios for both heart beats. Each scenario was re-run 20 times at 1%, 2%, 5% 10% and 20% noise levels. The MFS method was implemented by moving the torso nodes 40mm outward from the original surface and the heart nodes 10mm inward.

Inverse solutions were compared in terms of the conventional relative error and correlation coefficient measures [11].



(a) Front (b) Back

Figure 2. Front and rear views of the mesh with the large dots indicating electrode positions.

## 3. Results

As one particular instance, Figure 3 shows the potential distributions obtained at 205ms through a normal heart beat, with 1% measurement noise and the heart offset 10mm in the positive $x$ direction. Panel (a) shows the input distribution on 610 heart surface nodes and panel (b) shows a sub-sample of the same distribution on 114 nodes to enable comparison with the inverse solutions. The optimal BEM solution is shown in panel (c) and optimal MFS solution in panel (d). In this example, a lower relative error and higher correlation coefficient is obtained with the MFS method. Both BEM and MFS place the minimum of the distribution at the same point, but the maxima are at substantially different positions. Also, the potentials recovered from the MFS approach are overall lower than those obtained from the BEM approach.

Tables 1 and 2 present a summary of the performance of the regularisation parameter determination methods for the BEM and MFS approaches, respectively. Both tables show that the RGCV method most often produces the "better" (in the sense of lowest relative error and highest correlation coefficient) solution across the entire range of simulations considered, with the exception of the correlation coefficient for MFS and the ectopic beat where the GCV approach is slightly ahead. Again, from both tables it can be seen that the RGCV method more often produces the better solution for the normal beat than the ectopic beat. It also produces more minimum relative errors than maximum correlation coefficients for BEM. Finally, a comparison of these two tables shows that RGCV more often produces the better solution with MFS than with BEM.

Based on the arguments above, the RGCV method was chosen to compare inverse solutions from the BEM and MFS methods. The results are presented in Table 3, along with results obtained by using the optimal regularisation parameter with both BEM and MFS. In most cases the MFS solution can only determine the better solution in less than half the total number of simulations. Generally,

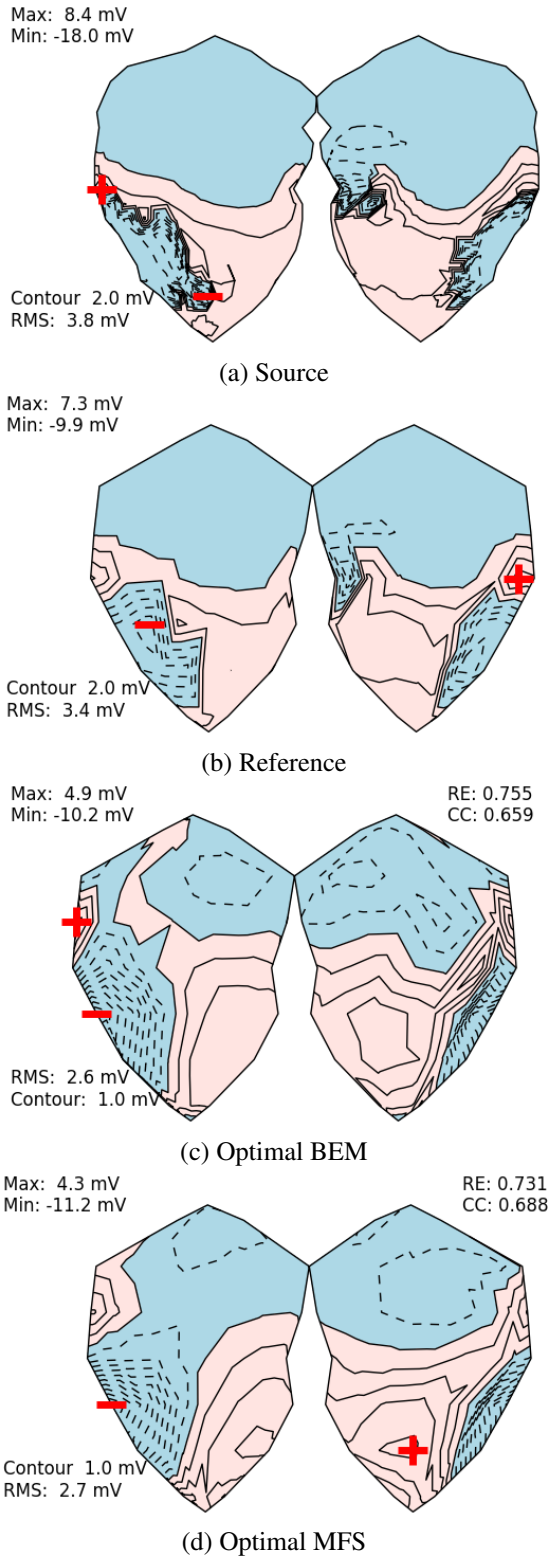



Figure 3. Optimal heart surface potential distributions obtained 205ms through a normal heart beat. Each panel shows the front and back views of the heart, the maximum and minimum potentials, the contour interval and the root mean square (RMS) value of the potential.

| | Rel. Error | | Corr. Coeff. | |
|---|---|---|---|---|
| | Normal | Ectopic | Normal | Ectopic |
| RGCV | 133 | 110 | 101 | 88 |
| CRESO | 26 | 27 | 36 | 39 |
| GCV | 10 | 11 | 26 | 12 |
| L-Curve | 56 | 85 | 74 | 83 |
| ZeroX | 55 | 47 | 43 | 58 |

Table 1. Number of times (out of 280) the indicated inversion method obtained the lowest relative error or highest correlation coefficient, respectively, for BEM.

| | Rel. Error | | Corr. Coeff. | |
|---|---|---|---|---|
| | Normal | Ectopic | Normal | Ectopic |
| RGCV | 146 | 131 | 166 | 113 |
| CRESO | 50 | 44 | 44 | 38 |
| GCV | 38 | 90 | 58 | 116 |
| L-Curve | 31 | 14 | 12 | 11 |
| ZeroX | 15 | 1 | — | 2 |

Table 2. Number of times (out of 280) the indicated inversion method obtained the lowest relative error or highest correlation coefficient, respectively, for MFS.

MFS yields more better solutions from the ectopic beat than from the normal beat. However, when the RGCV method is used with MFS, the proportion of better solutions increases, as compared with the optimal solutions.

## 4. Discussion

This study has compared the inverse solutions obtained from the MFS with those obtained from the BEM under several scenarios. While the MFS is more straightforward to implement, the BEM more often produces better solutions. In one particular example (Figure 3) MFS yields a lower relative error and higher correlation coefficient than BEM. However, it could be argued that the BEM solution is visually more accurate than the MFS solution.

One aspect of MFS that was not studied here was the creation of the new source points. This certainly does affect the accuracy of the solution method, and hence the quality of the inverse solutions.

It would appear that there is a trade-off here between ease of implementation and accuracy. Further comparison studies are certainly worthy of consideration.

| | Rel. Error | | Corr. Coeff. | |
|---|---|---|---|---|
| | Normal | Ectopic | Normal | Ectopic |
| Optimal | 89 | 129 | 109 | 117 |
| RGCV | 118 | 146 | 104 | 120 |

Table 3. Number of times (out of 280) MFS solution is "better" than BEM solution.

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